Exercise 59

Find the derivative of the function. Simplify where possible.

$$y = \arccos\left(\frac{b + a\cos x}{a + b\cos x}\right), \quad 0 \le x \le \pi,\ a > b > 0$$

Solution

Use the quotient rule, the chain rule, and the derivatives of the inverse trigonometric functions listed on page 214.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\arccos\left(\frac{b + a\cos x}{a + b\cos x}\right) \\
&= -\frac{1}{\sqrt{1 - \left(\frac{b+a\cos x}{a+b\cos x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{b + a\cos x}{a + b\cos x}\right) \\
&= -\frac{1}{\sqrt{1 - \frac{(b+a\cos x)^2}{(a+b\cos x)^2}}} \cdot \frac{\left[\frac{d}{dx}(b + a\cos x)\right](a + b\cos x) - (b + a\cos x)\left[\frac{d}{dx}(a + b\cos x)\right]}{(a + b\cos x)^2} \\
&= -\frac{1}{(a + b\cos x)\sqrt{1 - \frac{(b+a\cos x)^2}{(a+b\cos x)^2}}} \cdot \frac{(-a\sin x)(a + b\cos x) - (b + a\cos x)(-b\sin x)}{a + b\cos x} \\
&= -\frac{1}{\sqrt{(a + b\cos x)^2 - (b + a\cos x)^2}} \cdot \frac{-a^2\sin x - \cancel{ab\sin x\cos x} + b^2\sin x + \cancel{ab\sin x\cos x}}{a + b\cos x} \\
&= -\frac{1}{\sqrt{(a^2 + \cancel{2ab\cos x} + b^2\cos^2 x) - (b^2 + \cancel{2ab\cos x} + a^2\cos^2 x)}} \cdot \frac{(b^2 - a^2)\sin x}{a + b\cos x} \\
&= -\frac{1}{\sqrt{a^2 + b^2\cos^2 x - b^2 - a^2\cos^2 x}} \cdot \frac{(b^2 - a^2)\sin x}{a + b\cos x} \\
&= \frac{1}{\sqrt{(a^2 - b^2) + (b^2 - a^2)\cos^2 x}} \cdot \frac{(a^2 - b^2)\sin x}{a + b\cos x} \\
&= \frac{1}{\sqrt{(a^2 - b^2) - (a^2 - b^2)\cos^2 x}} \cdot \frac{(a^2 - b^2)\sin x}{a + b\cos x} \\
&= \frac{1}{\sqrt{(a^2 - b^2)(1 - \cos^2 x)}} \cdot \frac{(a^2 - b^2)\sin x}{a + b\cos x}
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{(a^2-b^2)\sin^2 x}} \cdot \frac{(a^2-b^2)\sin x}{a+b\cos x} \\ &= \frac{1}{\sqrt{a^2-b^2}\sin x} \cdot \frac{(a^2-b^2)\sin x}{a+b\cos x} \\ &= \frac{\sqrt{a^2-b^2}}{a+b\cos x}.\end{aligned}$$